Learning Times Tables Through Systematic Connections

Colin Foster

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Introduction

This short booklet offers a carefully-planned, systematic route to efficient learning of the times tables up to 12 × 12. The organising principle is the *multiplicative connections* between the different facts. Conceptually-oriented approaches to learning tables always emphasise the importance of connections. However, they typically proceed table by table (e.g., 2s, 10s, 5s, 4s, 3s, etc.), and this tends to promote *additive* rather than *multiplicative* thinking. For example, a learner struggling to find, say, 7 × 8 might be encouraged to arrive at this in a variety of ways from other (hopefully) known facts:

e.g. $7 \times 7 + 7$ $8 \times 8 - 8$ $10 \times 7 - 2 \times 7$ $10 \times 8 - 3 \times 8$

All of these calculations involve sophisticated addition/subtraction decisions (e.g., "To get from 8 × 8 to 7 × 8, do I subtract 8 or 7?") and rely on various other facts which the learner may also not yet have mastered. In some cases, the learner may only really trust skip-counting up from zero, where just one slip puts all the subsequent numbers out.

In the programme outlined in this booklet, the multiplication facts are experienced through their *multiplicative* connections and in an order that prioritises learning the highest-leverage facts first. Every fact has *one prioritised way* to connect it to what is already known, resulting in systematic, repeated retrieval practice of previous facts as new facts are established. For example, all of the facts that depend on 3 × 4 (i.e., 6 × 4, 3 × 8 and 6 × 8) provide opportunities to reinforce 3 × 4 as they build on it. Not only should this approach be more efficient, it should also lead to greater insight.

In this programme, there are 9 base facts (3^2, 4^2, 6^2, 7^2, 8^2, 12^2, 3 × 4, 3 × 7, 4 × 7) and all of the other facts can be derived 'easily' (with practice!) from these. Only 13 of the facts require calculation, which is always 'doubling' or 'doubling twice' – often in 'easy' situations in which there is no 'carry'. Some teachers or schools will judge that the 11-times and/or 12-times tables are not useful enough to warrant the effort needed to learn them, and so will stop at 10 × 10. In this case, that reduces the programme to just 8 base facts and 7 calculations.

Building facts on a small number of high-leverage base facts makes work on times tables truly mathematical, focused on connections rather than mere 'rote recall'. It should also be much more robust, because if a fact happens to be hazy or forgotten, it can always be built back quite quickly from known facts, and the techniques can be extended to calculations beyond the 1-to-12 multiplication tables.

The goal of this programme is not just for learners to end up confident about all 144 facts but to have a sense of the rich connected nature of the multiplication square and insights into the way in which mathematics builds and links in a multitude of exciting ways. One of the great benefits of an *efficient* scheme for learning the tables is that it should considerably reduce the pressure to *quickly* learn lots of new facts, and so allows time to work at learners' own pace and enjoy noticing things along the way.

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Step 1: Recognise what you already know!

Show learners the 12 × 12 square that we are aiming to know.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
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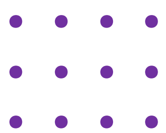
What do you notice? What mathematical questions could you ask?

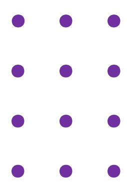
Viewing the entire square need not be daunting but exciting. It is much easier to learn it than it looks. *You already know more than you might think you know!*

1. Commutativity

Do you know that 3 × 4 = 4 × 3?

Even if a learner doesn't yet know *how much 3 × 4 is*, they can know this equality, by thinking visually about 3 rows of 4 counters and 4 rows of 3 counters:





They are the same *arrangement* of counters, if you just turn your head, so they must have the same *number* of counters.

Encourage learners to make rectangles (including squares) of counters and say what they see.

Commutativity means that almost half of the tables square is just a duplicate of the other half, so there is much less to learn than it first appears.

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2. The 1-times table, 2-times table, 5-times table and 10-times table

If learners don't already know these, then take time to learn them by exploring their patterns and repeating them. These tables don't so much need 'memorising' as appreciating. Once learners see their structure, they 'know' how they go. (The same is true for the 0-times table, not shown on the grid – be sure to include that!)

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Step 2: Learn the squares

The square numbers give a lot for a little. They take you right in close among the most 'difficult' of the multiplication facts. So, learning the squares early on is a high-leverage thing to do.

Learners already know 1^2, 2^2, 5^2 and 10^2, so there are 8 more squares to learn.

Make square arrays out of a variety of objects and encourage learners count how many there are in each. (In how many different ways can they efficiently count these arrays? *How can you find out how many there are without counting each one separately?*)

Practise the 12 squares until they know them well; spend a long time on this. They are lovely numbers to know. Enjoy them! Don't rush on past this stage. Even if learners never learn any more tables than these, then these are the most useful ones to have.

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There can be an early sense of achievement in knowing some of the big numbers (> 100) and in already reaching 12 × 12, which is 'the end'.

Step 3: Make the most of doubling

If learners know that 3 × 3 = 9, then 6 × 3 must be *twice as much* as this. This is a hard idea to get at first, because only *one* of the 3s is being doubled. Spend time on creating other examples of this idea and checking that it works.

Since they know their 2-times table, learners know that double-9 is 18. So they can now get 6 × 3 from 3 × 3.

They can do the same for 3 × 6: it must *also* be double-3^2.

Now they can do 3 × 6, 4 × 8 and 6 × 12 (and of course their reverses).

They can also do all of the 4-times table, because they can double the 2s.

We double the double-digit 2-times tables (> 10) by doubling the 1s, and doubling the 10s, and adding these together.

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Doubling is an important skill in its own right, so spend lots of time doubling everything in sight! Don't rush on past this stage until learners can confidently double any 2-digit number, even when there are 'carries' involved (i.e., when the 1s digit is more than 4).

Step 4: Learn how to do the 11s

We can learn the 11s now (or even sooner), because they are so easy.

This is because 11 lots of anything is the thing itself plus 10 lots of it, and we know the 10-times table.

The slightly trickier 11^2 fits this pattern, but is already known.

The only slightly awkward one left is 11 × 12, which also follows the same pattern, and can be found as 10 × 12 + 12.

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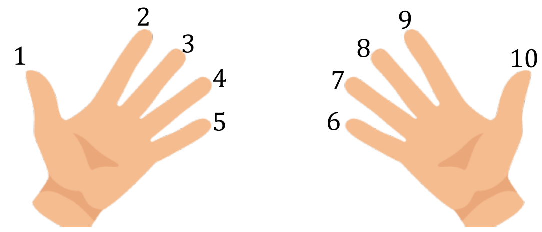
Now do lots of practice of all the coloured-in facts shown above. Remember to include the greens, as well as facts from the 0-times table (not shown on the grid). The challenge is for learners to switch from facts which they 'just know' to ones where other methods are needed, and become fluent switching between methods.

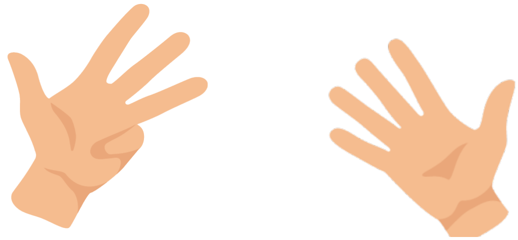
Step 5: Use your fingers to do the 9s

Perhaps you regard it as a cop-out to use fingers as a memory aid for the 9-times table? For me, this is not at all like those ‘memory tricks’ (e.g., 5, 6, 7, 8 for 56 = 7 × 8) because it is a general method that gives *all* of the 9-times table. With practice, learners will eventually not even need to ‘get out’ their fingers.

We hold out 10 fingers and number them from 1 to 10.



To find, say, **5** × 9, we fold down finger number **5** and read off the 4 fingers to the left of this as the 10s digit (40), and the 5 fingers on the right as the 1s digit (5), giving 45.



The reason this works is that $n \times 9$ is equal to $n \times 10 - n$. The 10s digit of the answer has to be 1 less than whatever n is (because we are subtracting a number smaller than 10 from $n \times 10$) and the 1s digit has to be n less than all 10 fingers, because we are subtracting n from a multiple of 10.

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Step 6: Learn 3 × 4, 3 × 7 and 4 × 7

Perhaps you want learners to learn *all* of the 3-times table, or they already know it?

Within this programme, the only 3-times tables that they *really* need are 3 × 4 and 3 × 7. So, in this step we learn these, along with 4 × 7.

Learners can already work out 3 × 4 from doubling 3 × 2, and 4 × 7 from doubling 2 × 7, but it is useful to 'just know' them, and 3 × 7 is also useful to have immediately available, because many other multiplication facts can be quickly obtained from them (see Step 7).

The ideal way for learners to reach the point where they 'just know' these is for you to keep asking them to calculate them. Eventually, they will find that, say for 4 × 7, they no longer need to 'go via 14', and instead they 'just know' that it is 28. Ideally, this will happen for many of the tables as time goes on, but the approach in this programme is intended to mean that, should any fact at any point slip away (or the learner feels unsure), it can always be recovered via others without too much trouble.

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Step 7: Get the rest through more doubling

Armed with our superpower of doubling, from 3 × 4 and 3 × 7 we can get a lot:

3 × 8, as double 3 × 4
6 × 8, as double double 3 × 4
6 × 7, as double 3 × 7
7 × 12, as double double 7 × 3

None of these doublings involves any ‘carries’, so they are not to hard to do mentally, with practice, even the ‘double doubles’. So these are good ones to do first.

You can also do:

9 × 12, as double 9 × 6
7 × 8, as double 7 × 4

And:

3 × 12, as double double 3 × 3
8 × 12, as double double 2 × 12

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‘Double double’ may sound hard, but if learners can double once then why not twice? It just takes practice.

And now they have all 144 facts available.

Summary

In this scheme there are 9 base facts which are learned:

$$3^2, 4^2, 6^2, 7^2, 8^2, 12^2, 3 \times 4, 3 \times 7, 4 \times 7$$

Then there are only 13 facts which require calculation:

3 × 6 = double 3 × 3	4 × 6 = double 2 × 6				
		6 × 7 = double 3 × 7			
3 × 8 = double 3 × 4	4 × 8 = double 4 × 4	6 × 8 = double double 3 × 4	7 × 8 = double 7 × 4		
3 × 12 = double double 3 × 3	4 × 12 = double 2 × 12	6 × 12 = double 6 × 6	7 × 12 = double double 7 × 3	8 × 12 = double double 2 × 12	9 × 12 = double 9 × 6

That completes all 144 facts.

And, if you are not teaching the 12-times table, then that reduces to just 8 base facts and 7 calculations.

The key to supporting the approach outlined in this booklet is in the prompts that you give when a learner is uncertain or stuck. For example, if a learner is stuck on 6 × 4, you might prompt (with increasing scaffolding):

- *What other fact can you get it from?*
- *Can you get it from 3 × 4?*
- *What do you need to do to 3 × 4 to make 6 × 4?*
- *What's changed from 3 × 4 to 6 × 4?*

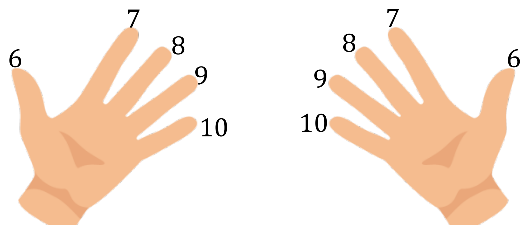
Initially, it can be helpful to be careful to offer the relevant fact 'the right way round'; i.e., here as 3 × 4, rather than 4 × 3, so that the order matches with 6 × 4. Later on, you might deliberately offer it the other way round, so that the learner has to mentally reverse it.

If all else fails...

If, for whatever reason, the programme outlined in this booklet doesn't work for you and your learners, there are other options. It is of course very rare nowadays not to be able to access a calculator or computer to do basic arithmetic. Even if these devices are not immediately to hand, learners always have their fingers, and it is possible to work out all the tables from 6-times upwards (the harder ones) on fingers, provided that they know up to their 5-times tables.

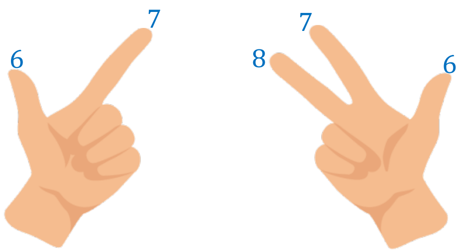
Here is how to do it.

Starting from 6, number your fingers on both hands, like this:



(Note that this numbering is different from the numbering used for the 9-times table in Step 5. If you are going to teach this method, then there is perhaps no need to teach the 9-times table method as well, and this avoids potential confusion.)

Suppose you want to work out 7 × 8.
Put out the fingers **up to 7** on one hand and **up to 8** on the other hand:



Add up the **number of fingers** on each hand that are **out**:

$$2 \text{ on the left hand} + 3 \text{ on the right hand} = 5$$

That's the 10s digit in the answer.

This will never go higher than 5 × 5 because there are a maximum of 5 fingers on each hand.

Multiply the number of fingers on each hand that are **not out**:

$$3 \text{ on the left hand} \times 2 \text{ on the right hand} = 6$$

That's the units digit in the answer.

So,

$$7 \times 8 = 56$$

Learners can get very quick at doing this, eventually just 'feeling' or visualising their fingers, rather than having to 'get their fingers out'.

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